

Penrose Diagram for a Transient Black Hole

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Abstract

A Penrose diagram is constructed for a spatially coherent black hole that smoothly begins an accretion, then excretes symmetrically as measured by a distant observer, with the initial and final states described by a metric of Minkowski form. Coordinate curves on the diagram are computationally derived. Causal relationships between space-time regions are briefly discussed. The life cycle of the black hole demonstrably leaves asymptotic observers in an unaltered Minkowski space-time of uniform conformal scale.

1 Introduction

Black holes continue to be objects of considerable analytic and observational interest. Because of expected phenomena such as Hawking radiation and holography, black holes provide meaningful systems connecting quantum behaviors to geometrodynamics. However, that very coupling to quantum processes implies that descriptions of the space-time should qualitatively differ from classical static systems. Such changes in description are needed to model dynamic space-times consistent with quantum measurement constraints.

One of the most interesting features of quantum systems is the space-like coherence of an entangled state. Motivated by the need to more directly describe such systems within a geometrically dynamic system, coordinates describing a spatially coherent geometry have been developed[1, 2]. The metric form utilizes non-orthogonal coordinates inspired by the river model of black holes[3, 4]. Fixed temporal coordinate curves remain space-like surfaces throughout such dynamic geometries, corresponding to the times measured by distant, inertial observers. In addition, the dynamic horizon and radial mass scale parameterized using this time have non-singular curvature and coordinate representations. These qualitative differences in the nearby environment of the dynamic horizon makes examinations of quantum behaviors more straightforward.

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2 Form of the Metric and Conformal Coordinates

2.1 Form of the metric

The radially dynamic space-time metric for a spatially coherent, spherically symmetric system will be assumed to take the form[1, 4, 5]

$$ds^2 = - \left(1 - \frac{R_M(ct, r)}{r} \right) (dct)^2 + 2 \sqrt{\frac{R_M(ct, r)}{r}} dct dr + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (2.1)$$

In this equation, a finite radial mass scale $R_M(ct, r) \equiv 2G_N M(ct, r)/c^2$ is the length scale of the mass-energy content of the geometry, which is given by the Schwarzschild radius for a static geometry. The metric takes the form of a Minkowski space-time both asymptotically ($r \rightarrow \infty$) as well as when the radial mass scale vanishes ($R_M(ct, r) \rightarrow 0$). Therefore, the temporal and radial coordinates are those of an observer far from the black hole. The radial coordinate also provides the length scale for local tangential distances and areas. As can be seen from the form of the metric Eqn. 2.1, at the surface instantaneously defined by the radial mass scale $R_M(ct, r)$, that fixed radial coordinate curve labeled r transforms from being time-like external to this scale to being space-like internal to this scale. Fixed temporal coordinate curves are always spacelike, making generic (non-orthogonal) geometries of this form convenient for explorations of quantum behaviors. This metric has been examined for uniformly accreting[6] and excreting[7, 8] black holes in previous articles.

For a transient black hole initiating at ct_o , and terminating at ct_f the outgoing light-like surface defining the horizon R_H is given by the null surface of the metric Eqn. 2.1 which vanishes as the decreasing radial mass scale vanishes at ct_f . This surface satisfies the general equation for outgoing null geodesics,

$$\dot{r}_\gamma = 1 - \sqrt{\frac{R_M}{r_\gamma}} \quad , \quad r_\gamma(ct) = \frac{R_M(ct, r_\gamma(ct))}{(1 - \dot{r}_\gamma(ct))^2}. \quad (2.2)$$

Several points of interest directly follow from this equation and the form of the metric:

- The radial mass scale falls interior to the horizon while the horizon is expanding $\dot{R}_H > 0$, and exterior to the horizon while it is contracting $\dot{R}_H < 0$. It should be noted that the transition time from mass accretion to mass excretion does not generally coincide with that from an expanding to contracting horizon.
- Since outgoing photons would be momentarily stationary as they cross the radial mass scale $\dot{r}_\gamma = 0$, there can be no observers with stationary radial co-

ordinate (fiducial observers) with $r \leq R_M$. The radial mass scale represents the static limit for radial motions in this geometry.

- From the second form of the equation, it is clear that a finite horizon scale vanishes when the radial mass scale vanishes, as long as $\dot{R}_H < 1$. Of course, the radial mass scale *must* vanish if $\dot{r}_\gamma = 1$.
- There are trapped outgoing light-like surfaces which initially propagate with $\dot{r}_\gamma > 0$ through non-vanishing radial mass scale R_M with radial coordinates that vanish as $ct \rightarrow ct_o$, as well as at some final time $ct < ct_f$ on the singularity. These trajectories will be demonstrated later in Figure 3b. This means that $R_M/r_\gamma < 1$ for times infinitesimally close to ct_o .
- Since distinct outgoing light-like trajectories can initiate on the surface ($ct = ct_o, r_\gamma = 0$), for spatially coherent dynamic geometries the initiation of the (possible) singularity should be represented by an ingoing light-like surface. The curve $r = 0$ should transform through this surface from time-like to space-like.

Radial trajectories in this geometry have 4-velocity components that satisfy

$$u^r = -u^0 \sqrt{\frac{R_M}{r}} \pm \sqrt{(u^0)^2 - \Theta_m} \quad , \quad \Theta_m \equiv \begin{cases} 1 & m \neq 0 \\ 0 & m = 0 \end{cases} \quad , \quad (2.3)$$

where the + sign signifies outgoing trajectories, and the - sign signifies ingoing trajectories. It should be noted that for massive systems, trajectories $u^0 = 1$ are neither ingoing nor outgoing, and they represent what have been referred to as *geometrically stationary* trajectories[8]. Trajectories with 4-velocity components satisfying $u^0 = 1$, $u^r = -\sqrt{\frac{R_M}{r}}$ satisfy geodesic equations for massive gravitating systems which share proper time with asymptotic observers, $dt = d\tau$.

The static form of this geometry $\dot{R}_M = 0$ satisfies the same form of dominant energy conditions as does a generalization of orthogonal Schwarzschild space-time with a spherically symmetric static mass distribution. A spatially coherent system whose temporal variations are sufficiently slow can likewise satisfy these conditions. One can therefore directly construct models of physical systems whose only violations of energy conditions are consistent with those due to quantum effects.

Since the large scale causal structure of the space-time represented by the Penrose diagram should not depend sensitively on the form of the radial mass scale, one can choose a relatively simple form for its functional dependence. For the present calculation, the radial mass scale will be taken as a time-dependent form given by the “bump function”

$$R_M(ct) \equiv R_{max} \exp\left(\frac{(ct_f)^2}{(ct)^2 - (ct_f)^2}\right) \quad (2.4)$$

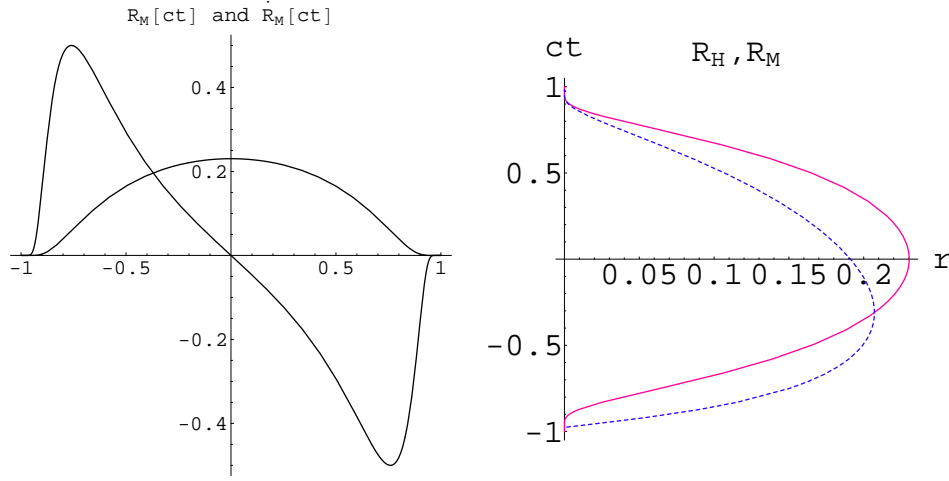


Figure 1: Left: Bump function and its derivative. Right: Horizon (dotted, blue) and radial mass scale (solid, red).

during the period $ct_o = -ct_f \leq ct \leq ct_f$. All curvature components generated by this metric are non-singular away from the physical singularity $r = 0$ as seen from the Ricci scalar

$$\mathcal{R} = 3 \left(\frac{ct_f}{(ct)^2 - (ct_f)^2} \right)^2 \frac{ct}{r} \sqrt{\frac{R_M(ct)}{r}}. \quad (2.5)$$

This means that no observer measures singular curvatures at the horizon, radial mass scale, or transition times of the geometry.

2.2 Development of conformal coordinates

The technique used to generate the conformal diagram relied only upon constructing light-like surfaces for the metric Eqn. 2.1. The conformal coordinates will be labeled (v, u) . For ingoing null geodesics, the required equation takes the form

$$\dot{r}_v = -1 - \sqrt{\frac{R_M}{r_v}}, \quad (2.6)$$

while outgoing null geodesics satisfy

$$\dot{r}_u = 1 - \sqrt{\frac{R_M}{r_u}}. \quad (2.7)$$

Ingoing light-like trajectories labeled by v can access all regions of space-time through a past Minkowski space-time correspondence, initiating on past light-like infinity $scri^-$. For the chosen geometry, ingoing light-like trajectories are demonstrated in Figure 2.

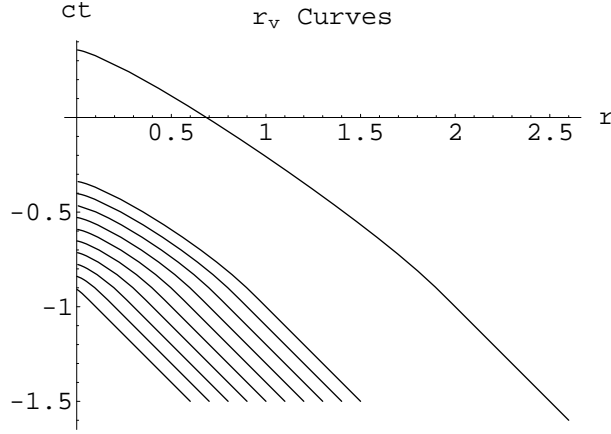


Figure 2: Ingoing light-like trajectories, labeled by conformal coordinate v .

Likewise, for the region exterior to the horizon, outgoing light-like trajectories labeled by u passing through any point have a future Minkowski correspondence, terminating on future light-like infinity $scri+$. One only needs to develop labels u for outgoing light-like trajectories in the interior of the horizon. The functional representation of the singularity on the conformal diagram $u = u_{r=0}(v)$ defines an analytic extension that parameterizes these trajectories in terms of their termination values v on the singularity. Several outgoing light-like trajectories for the given geometry are demonstrated in Figure 3. The trajectories of outgoing photons near the horizon are of particular interest. An observer with a fixed radial coordinate external to the radial mass scale and horizon will not observe photons emitted near the horizon until after the singularity has vanished. This means that an object falling through the horizon will not be observed to cross the horizon until after the black hole is no longer present. The moment just prior to the observation of the final evaporation of the singularity is directly seen to provide all observable information on matter that fell through the horizon during its period of transience. Coherence relationships with exterior constituents need not be disrupted as infalling constituents cross the horizon.

The curves representing the radial centers $r = 0$ of the initial and final state Minkowski space-times of the transient black hole should not be expected to be co-linear vertical lines. Since light-like trajectories get shifted relative to those in flat space-time due to gravitational attraction, the conformal Minkowski coordinates $(v = ct + r + v_s, u = ct - r + u_s)$ are relatively shifted by factors (v_s, u_s) that are fixed on the particular trajectory. For the present calculations, the label v will be taken from the past Minkowski initial state surface $v = ct_o + r_o$, while the exterior labels u will be taken at the future Minkowski final state surface $u = ct_f - r_f$, providing boundary conditions for the differential equations 2.6 and

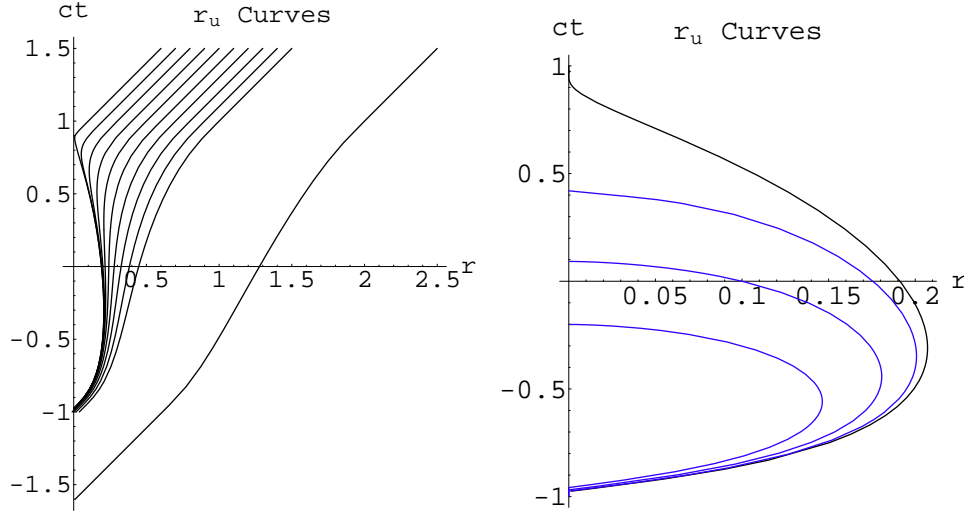


Figure 3: Exterior (a) and interior (b) outgoing light-like trajectories, each labeled by the value of conformal coordinate u .

2.7. These labels can be directly observed from the surface $ct_o = -1$ on Fig. 2 and $ct_f = +1$ on Fig. 3a.

3 Penrose Diagram of the Transient Black Hole

Space-time diagrams can be quite useful for visualizing the dynamic relationships in a given geometry. Penrose diagrams are convenient for examining the large-scale causal structure of a geometry because of the following properties:

- ... Penrose diagrams map the entire space-time onto a single finite page, and
- ... Penrose diagrams are conformal diagrams, i.e., they preserve the slope of light-like trajectories at \pm unity.

Light-like surfaces define the boundaries of causal regions in space-time, so the causal structure of the geometry can be directly observed from any conformal diagram, and potential causal relationships between events can be immediately ascertained. Minkowski coordinates (ct, r) of flat space-time are already conformal coordinates. However, while $R_M \neq 0$ for the metric 2.1, null geodesics do not have unity slope for coordinates (ct, r) as seen in Figures 2 and 3. Therefore, the light-cone conformal coordinates (v, u) will be used to construct the Penrose diagram for the transient black hole.

3.1 Procedure for constructing the Penrose diagram

The conformal diagrams presented use hyperbolic tangents of a scaled multiple of the conformal coordinates (v, u) to map the infinite domain of those conformal coordinates onto a finite region. The coordinate Y_{\uparrow} labeling the vertical axis takes the form $(\tanh(\frac{v}{scale}) + \tanh(\frac{u}{scale}))/2$ and the coordinate Y_{\rightarrow} labeling the horizontal axis takes the form $(\tanh(\frac{v}{scale}) - \tanh(\frac{u}{scale}))/2$, so the diagram has its domain and range bounded by ± 1 . In the interior region $r < R_H(ct)$, the analytic continuation must be chosen to generate a representation for the singularity that is space-like (except perhaps at the endpoints). Since the entire coordinate system has been generated using light-like trajectories from Eqns. 2.7 and 2.6, the slopes of *any* outgoing/ingoing light-like radial trajectories on the diagram will automatically be ± 1 . The overall causal structure of the transient black hole is demonstrated in Figure 4. The figure demonstrates the initial and final time volumes as green

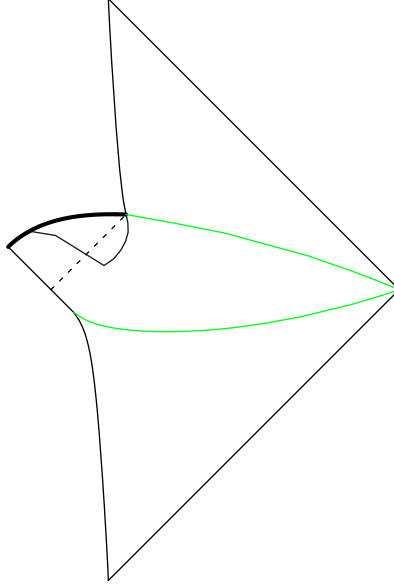


Figure 4: Horizon, radial mass scale, singularity, and overall causal structure of the black hole.

horizontal curves terminating at the right corner of the diagram. Prior to the initial volume state and subsequent to the final volume state, the space-time is that of Minkowski. The diagram is bounded on the right by past light-like infinity *skri minus* (bottom right) and future light-like infinity *skri plus* (top right).

The diagram is bounded on the left by the center $r = 0$, which is space-like during the period of transience of the black hole, light-like during the initiation of the singularity, and otherwise time-like. The light-like surface initiating the

singularity has coordinates $(ct = -1unit, r = 0)$. This ingoing light-like surface expands the conformal space-time[6] in a manner differing from standard textbook treatments[9, 10]. The horizon is the dotted surface with unit positive slope, which is crossed once by the solid curve representing the radial mass scale. The interior region is left of the horizon, bounded from above by the singularity. A more detailed description will be given in the next section.

The initial and final geometries are Minkowski space-times for $|ct| \geq 1unit$. The bump function smoothly transitions the dynamic metric in Eq. 2.1 into static forms. The radial coordinates must smoothly match across the transition volumes $|ct| = 1unit$, since $4\pi r^2$ measures the area of any sphere of radial coordinate r for both local conformal coordinates (v, u, θ, ϕ) as well as those of the asymptotic observer (ct, r, θ, ϕ) .

3.2 Features of the Penrose diagram

The Penrose diagram in Figure 5 demonstrates the expected global structure of this spherically symmetric, spatially coherent black hole that smoothly varies the radial mass scale $R_M(ct)$ with respect to the distant observer's time coordinate ct . The parameters were chosen such that the maximum rate of mass accretion/excretion is $|\dot{R}_M| \leq 0.5$, yielding a maximum radial mass scale $R_M(0) \cong 0.23units$, which gives a maximum horizon scale of $R_H(-0.31units) \cong 0.21units$. In the diagram, the red curves that are time-like (vertical) in the right hand regions represent curves of constant r , originally graded from $r = 0$ in hundredths, tenths, then in units of the chosen scale. The curves of constant radial coordinate r all originate at the bottom corner of the diagram representing $t = -\infty$, and terminate at the uppermost corner representing $t = +\infty$. The green curves, which, away from initiation or termination surfaces, are everywhere space-like (horizontal), represent curves of constant ct graded in units of the given scale. All constant ct curves originate on the curve $r = 0$ and terminate at the far right corner of the diagram representing $r = \infty$. The various $ct = constant$ and $r = constant$ curves each intersect at only one point on the diagram (except for the initiation of the singularity, and light-like infinities). The light-like bounding curves $r = \infty$ on the right are those of a fixed scale Minkowski space-time.

One should note that at $ct = -1unit$, the singularity develops via an ingoing light-like transition $(ct = ct_o, r = 0)$, expanding the available conformal space-time. This new region, represented as the expanded area in the middle left portion of Figure 5, contains most of the significant features of the black hole. The curve $r = 0$, which bounds the diagram on the left, is initially a time-like trajectory bounding the initial Minkowski space-time. However, due to relative parameter shifts because of the transient black hole, this curve is not represented by a vertical line on this diagram. Rather, this curve smoothly joins the curve $ct = ct_o$ at the transition. As the singularity forms, the curve $r = 0$ undergoes a light-like

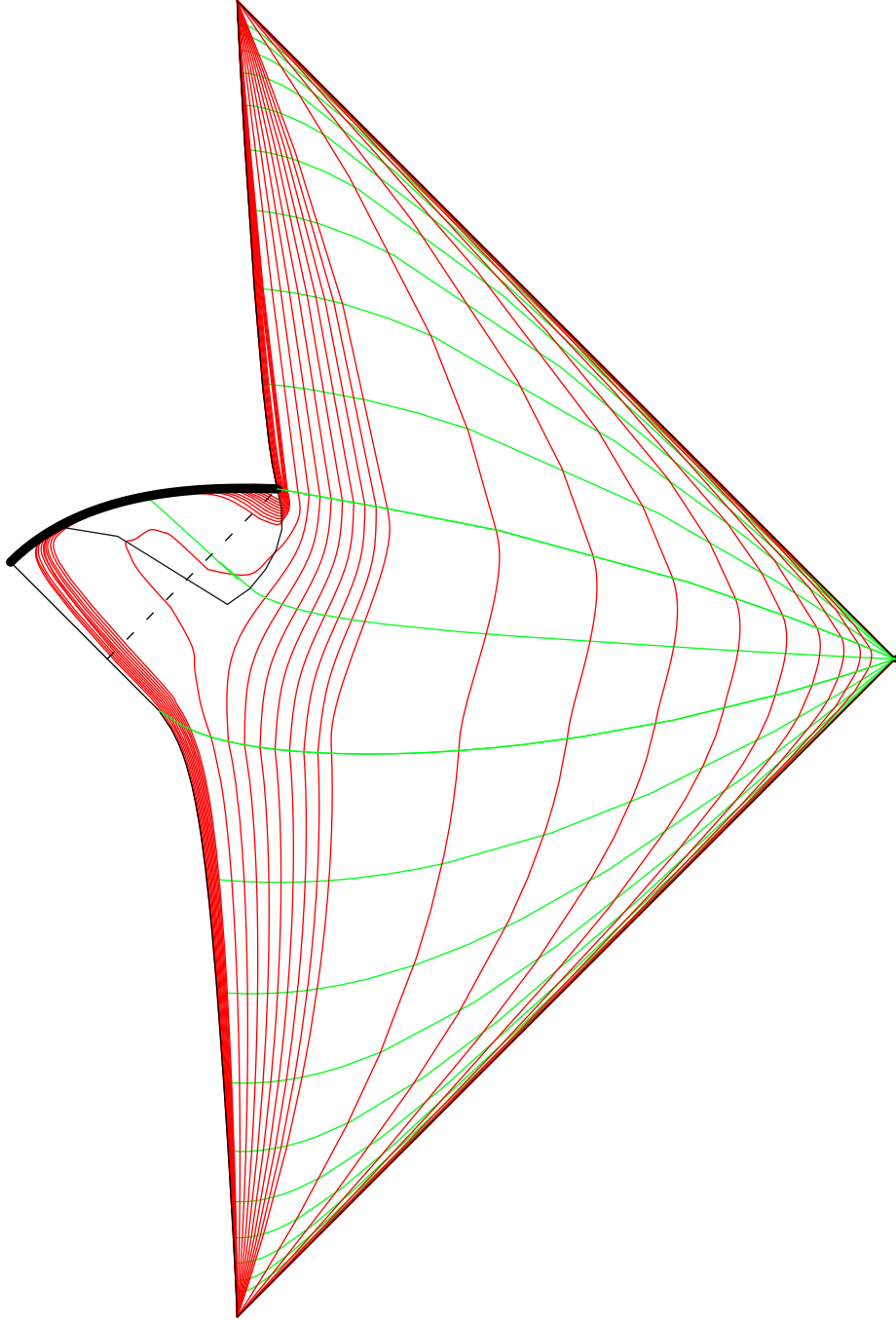


Figure 5: Penrose diagram for a black hole that symmetrically accretes from zero mass at $ct = -1$ then excretes to zero mass at $ct = +1$. Red curves (running vertically in the exterior right hand region) represent curves of constant r . The green curves represent curves of constant ct . The dashed line represents the horizon, and the solid, bold space-like curve represents the singularity.

transition to become a space-like trajectory bounding the upper interior region of the black hole from above. As previously mentioned, the form of this space-like surface and the transition to it *defines* the analytic continuation of correspondence with the exterior coordinates. As the singularity vanishes, the curve $r = 0$ bounds the final Minkowski space-time from the left, again skewed from the vertical. The radial mass scale is indicated by the solid dark gray curve initiating at the far left corner of the diagram, crossing the horizon R_H near $t = 0$, then terminating as the singularity vanishes. As can be seen from Eqn. 2.2 the radial mass scale $R_M(ct)$ lies within the horizon during growth of the horizon and outside of the horizon during shrinking of the horizon[1]. Radial coordinate curves are seen to transition from time-like to space-like as they cross the radial mass scale (as expected from the metric Eqn. 2.1 for fixed radial coordinate $dr = 0$). As previously noted, outgoing light-like trajectories Eqn. 2.7 are momentarily stationary in the radial coordinate $\dot{r}_u = 0$ at the radial mass scale.

The dynamic horizon is represented by the dashed diagonal line in Figure 5, terminating as the space-time reaches its final Minkowski form. It is noteworthy that the horizon lies completely within the expanded region of space-time to the left of this Minkowski geometry. The black hole horizon has a finite temporal duration; unlike the case for a static Schwarzschild black hole, both temporal and radial coordinate curves are seen to cross this surface, i.e. the horizon $R_H(ct)$ is *not* a $t = \infty$ surface.

3.3 Causal regions

Penrose diagrams are most convenient for exploring the large-scale causal structure of the space-time. For the transient black hole, various causal regions, are represented in Figure 6. The boundaries of the regions are shown as dotted light-like surfaces on the Penrose diagram. It is of considerable interest to examine causal relationships with the region interior to the horizon labeled IH. In particular, one might note that events in this region can share no direct causal relationship with events in the future region F (i.e., events in these regions share neither time-like, light-like, or space-like relationships). This means that even a quantum entangled relationship cannot be established between disparate events within these regions. However, a space-like correlated event associated with the interior can be a cause to an event in future region F. The interior region *can* have space-like entanglements (coherence) with systems in the near horizon (NH), exterior (E), black hole causal past (BH-), and black hole causal future (BH+) regions. Boundary conditions for any quantum systems with space-like coherence with the interior should be physically consistent across these regions of the space-time. Explicitly, interior entanglements cannot have boundary conditions within the future region F.

Other causal relationships are indicated in Fig. 7. The possible relationships

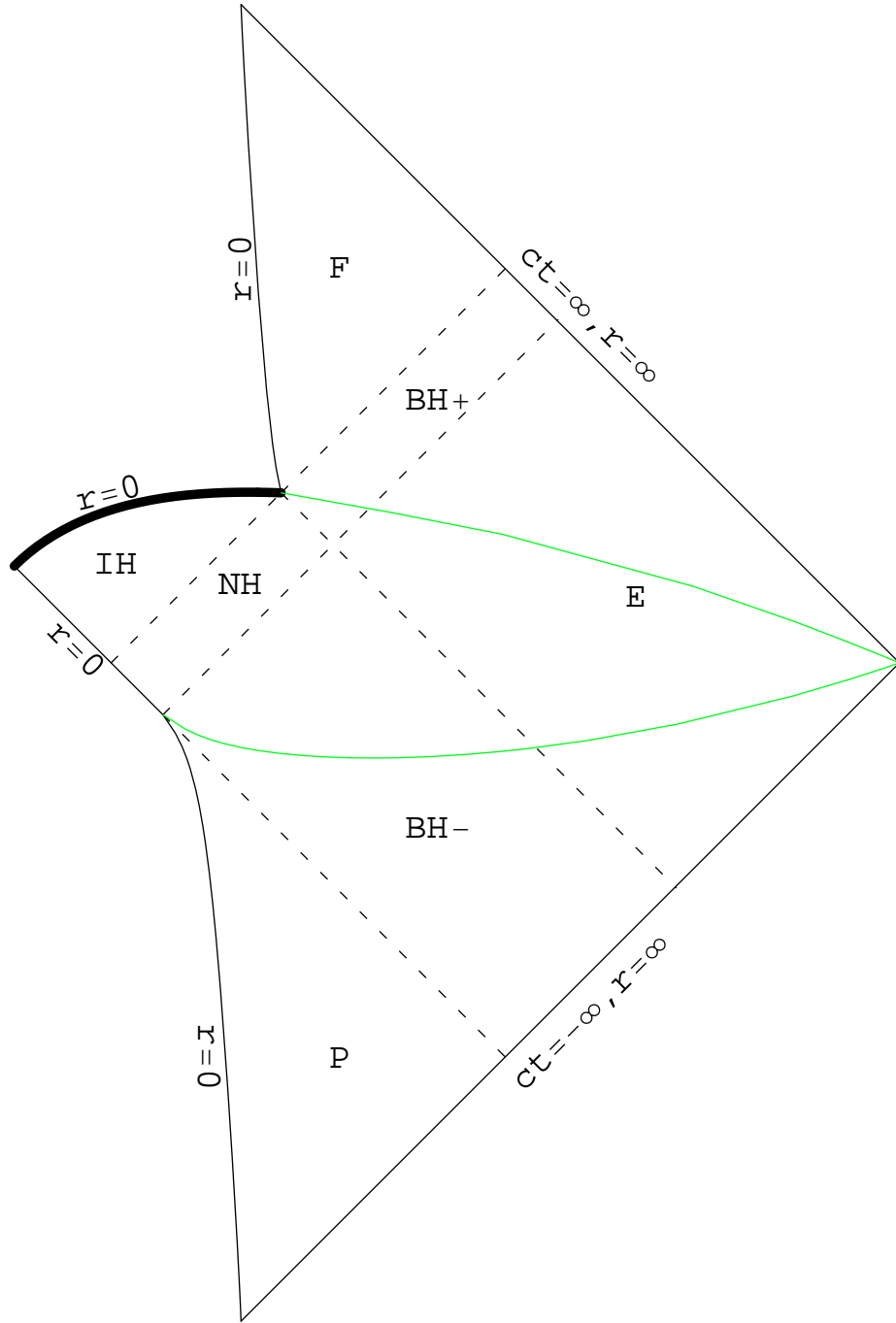


Figure 6: Causal regions include interior to the horizon IH, near horizon NH, exterior E, causal past BH-, causal future BH+, past P, and future F.

	IH	NH	E	F	P	BH-	BH+
IH	-	Effect/S-L	Spacelike	None	Effect	Effect/S-L	Spacelike
NH	Cause/S-L	-	Spacelike	Cause	Effect	Effect/S-L	Cause/S-L
E	Spacelike	Spacelike	-	Cause/S-L	Effect/S-L	Effect/S-L	Cause/S-L
F	None	Effect	Effect/S-L	-	Effect	Effect	Effect/S-L
P	Cause	Cause	Cause/S-L	Cause	-	Cause/S-L	Cause
BH-	Cause/S-L	Cause/S-L	Cause/S-L	Cause	Effect/S-L	-	Cause
BH+	Spacelike	Effect/S-L	Effect/S-L	Cause/S-L	Effect	Effect	-

Figure 7: Causal relations. Events in a region labeled by a given row possibly relate to those in a region labeled by a given column as indicated.

are indicated as either cause, effect, space-like (S-L), or none. Cause/effect relationships must be either (past/future) time-like or light-like. These relationships are completely consistent with those reported earlier in reference [11].

4 Conclusions and Discussion

Generic spherically symmetric, dynamic geometries with mass scale $M = M(ct, r)$ can be used to model dynamic physical systems. A spatially coherent dynamic black hole has been shown to maintain space-like, fixed time volumes throughout the global geometry, while generating a space-like surface $r = 0$ and an outgoing light-like horizon during its period of generating significant curvature. Parametric descriptions of both the radial mass scale and the horizon are non-singular in the coordinates (ct, r) , as are scalar forms of physical curvatures near these trajectories. The radial mass scale does not coincide with the light-like horizon.

The large-scale geometry of a transient black hole has been seen to have negligible effect upon the scales of distant observers. The past and future light-like infinities of the black hole are the same as those of Minkowski space-time. The onset of the singularity and a finite mass scale expands the previously low-curvature conformal space through an ingoing light-like transition, concurrently expanding the large-scale structure of the space-time.

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